Electromagnetic 2-forms on space-time

M. Grigorescu

Two field 2-forms on the space-time manifold, in a relationship of duality, are presented and included in the extended phase-space structure used to describe relativistic particles having both electric and magnetic charges. By exterior derivatives, these forms yield the two groups of Maxwell equations, while specific integrality conditions ensure magnetic monopole or electric charge quantization. Some properties of the common characteristic vector of the dual 2-forms are discussed. It is shown that the coupled energy-density continuity equation and the eikonal equation represent a classical, infinite-dimensional Hamiltonian system.

1

1 Introduction

The geometric description of the electromagnetic field in terms of a connection form on the space-time manifold, related to the variation of length during parallel transport, was considered in [1]. This connection form provides the Lorentz force, as it modifies the symplectic potential of the phase-space for an electric charge. Its curvature is a 2-form related to the first group of Maxwell equations, and Dirac's quantization condition for the magnetic monopole charge [2, 3]. A second 2-form can also be defined by a duality relationship [4, 5], and used to express the second group of Maxwell equations.

This work presents some properties of the 2-forms associated with the electromagnetic field, and their relevance for particle and photon dynamics. As these forms contain time as a coordinate rather than as a parameter, in Section 2 the motion of a relativistic electric charge is described in terms of a Hamiltonian vector field on the cotangent bundle of the space-time manifold (the extended phase-space). The case of a particle having also a magnetic charge is considered in Appendix 1. The second 2-form and the Maxwell equations are presented in Section 3. The common characteristic vector of the 2-forms is studied in Section 4. It is shown that the photon dynamics in a transparent medium can be described as Hamiltonian flow of classical particles, with density and phase as canonically conjugate variables. Conclusions are summarized in Section 5.

2 Relativistic charge in extended phase-space

The canonical coordinates $q^e \equiv (q_0, \mathbf{q})$ and $p^e \equiv (p_0, \mathbf{p})$ on the extended phase-space $M^e \equiv \mathsf{R}^8$ of a relativistic particle consist of the canonical coordinates $\mathbf{q} = (q_1, q_2, q_3)$, $\mathbf{p} = (p_1, p_2, p_3)$ on the usual phase-space $M \equiv \mathsf{R}^6$, and (q_0, p_0), supposed to be linear functions of time and energy, $q_0 = ct$, respectively $p_0 = -\mathcal{E}/c$, where c is a dimensional constant, identified with the speed of light in vacuum [6]. As the convention of rising/lowering vector indices by the Lorentz metric tensor will not be applied, here and in the following all vector indices are subscripted, and all components are the physical ones. Let u denote the "universal time" parameter along the trajectories on M^e, $d_u f \equiv df/du \equiv f'$ the derivative of f with respect to u, and X_{H^e} the

Hamiltonian vector field

$$X_{H^e} = \sum_{\mu=0}^{3} q'_\mu \partial_\mu + p'_\mu \partial_{p_\mu} \quad , \tag{1}$$

$\partial_\mu \equiv \partial/\partial q_\mu$, $\partial_{p_\mu} \equiv \partial/\partial p_\mu$, defined on M^e by

$$i_{X_{H^e}} \omega_0^e = dH^e \quad , \tag{2}$$

with the canonical symplectic form $\omega_0^e = -d\theta_0$,

$$\theta_0 = \sum_{\mu=0}^{3} p_\mu dq_\mu \quad . \tag{3}$$

A free relativistic massive particle can be described using the extended Hamiltonian[1]

$$H_0^e = -c\sqrt{p_0^2 - \mathbf{p}^2} \quad , \tag{4}$$

so that (2) provides the equations of motion

$$q'_0 = -c\frac{p_0}{\sqrt{p_0^2 - \mathbf{p}^2}} \quad , \quad p'_0 = 0 \tag{5}$$

$$\mathbf{q}' = c\frac{\mathbf{p}}{\sqrt{p_0^2 - \mathbf{p}^2}} \quad , \quad \mathbf{p}' = 0 \quad . \tag{6}$$

The usual veocity is $\mathbf{v} = c\mathbf{q}'/q'_0 = -c\mathbf{p}/p_0$, and the invariant value of $H_0^e = -m_0 c^2$ defines the rest mass.

For an electric charge e, the coupling to the vector and scalar potentials of the electromagnetic field $\mathbf{A} = (A_1, A_2, A_3)$, respectively V, can be introduced by a local shift between mechanical and canonical momentum, so that ω_0^e becomes

$$\omega^e = -d\sum_{\mu=0}^{3} (p_\mu + \frac{e}{c}A_\mu)dq_\mu \quad ,$$

where $A_0 = -\mathsf{V}$. Thus $\omega^e = \omega_0^e + e\omega_f/c$ contains beside ω_0^e the field 2-form $\omega_f = -d\theta_f$,

$$\theta_f = \sum_{\mu=0}^{3} A_\mu dq_\mu = \mathbf{A} \cdot d\mathbf{q} + A_0 dq_0 \quad . \tag{7}$$

---

[1] considered previously in [3], p. 43, and independently in [7].

The electric and magnetic fields $\mathbf{E} = -\partial_0 \mathbf{A} - \nabla \mathsf{V}$, and $\mathbf{B} = \nabla \times \mathbf{A}$, therefore appear as coefficients of the 2-form ω_f on the space-time manifold R^4 [1]

$$\omega_f = -\sum_{\mu < \nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) dq_\mu \wedge dq_\nu \tag{8}$$

$$= -\mathbf{B} \cdot d\mathbf{S} + \mathbf{E} \cdot dq_0 \wedge d\mathbf{q} \ ,$$

where $dS_1 = dq_2 \wedge dq_3$, $dS_2 = -dq_1 \wedge dq_3$, $dS_3 = dq_1 \wedge dq_2$. The elements $[\omega_f]_{\mu\nu} = -(\partial_\mu A_\nu - \partial_\nu A_\mu)$ can also be represented in the matrix form

$$[\omega_f] = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix} \ . \tag{9}$$

In the presence of the field, the equations of motion defined by $i_{X_{H^e}} \omega^e = dH^e$ are

$$q_0' = -c \frac{p_0}{\sqrt{p_0^2 - \mathbf{p}^2}} \ , \quad p_0' = -\frac{e}{c} \mathbf{E} \cdot \mathbf{q}' \ , \tag{10}$$

$$\mathbf{q}' = c \frac{\mathbf{p}}{\sqrt{p_0^2 - \mathbf{p}^2}} \ , \quad \mathbf{p}' = \frac{e}{c} (\mathbf{q}' \times \mathbf{B} + q_0' \mathbf{E}) \ . \tag{11}$$

Denoting by $\dot{\mathbf{q}} \equiv \mathbf{v} = c\mathbf{q}'/q_0'$, $\dot{\mathbf{p}} \equiv c\mathbf{p}'/q_0'$, $\dot{\mathcal{E}} \equiv -c^2 p_0'/q_0'$ the usual derivatives of $\mathbf{q}$, $\mathbf{p}$, $\mathcal{E}$ with respect to the time $t = q_0/c$, these equations yield

$$\dot{\mathbf{p}} = \frac{e}{c} \mathbf{v} \times \mathbf{B} + e\mathbf{E} \ , \quad \dot{\mathcal{E}} = e\mathbf{E} \cdot \mathbf{v} \ , \tag{12}$$

with $\mathbf{v} = \mathbf{p}c^2/\mathcal{E}$.

3 The Maxwell equations

The field 2-form ω_f has an associated dual ω_f^*, which can be defined by

$$\omega_f^* = \sum_{\mu < \nu, \alpha, \beta} (-\eta)^{\delta_{\alpha 0} + \delta_{\beta 0}} \epsilon_{\alpha\beta\mu\nu} \partial_\alpha A_\beta dq_\mu \wedge dq_\nu = \eta \mathbf{E} \cdot d\mathbf{S} + \mathbf{B} \cdot dq_0 \wedge d\mathbf{q} \ , \tag{13}$$

where[2] $\eta = \epsilon_r \mu_r$, ϵ_r, μ_r denote the relative dielectric permittivity and magnetic permeability coefficients, $\delta_{\alpha\beta}$ is the Kronecker symbol, and $\epsilon_{\alpha\beta\mu\nu}$ is the

---

[2] $\sqrt{\eta}$ is the refractive index of the medium, presumed to be a positive constant. Though, metamaterials with negative refractive index have also been obtained [8].

4

unit tensor ($\epsilon_{0123} = 1$), fully antisymmetric to the permutations of the four indices. The elements $[\omega_f^*]_{\mu\nu}$ can be represented in matrix form as

$$[\omega_f^*] = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & \eta E_3 & -\eta E_2 \\ -B_2 & -\eta E_3 & 0 & \eta E_1 \\ -B_3 & \eta E_2 & -\eta E_1 & 0 \end{bmatrix} . \tag{14}$$

If $\lambda \in SO(1,3)^*$ is a Lorentz transformation, $\lambda^T \hat{g} \lambda = \hat{g}$, $\hat{g} = diag[-1, 1, 1, 1]$,

$$\tilde{q}_\mu = \sum_{\nu=0}^{3} \lambda_{\mu\nu} q_\nu \quad , \quad \tilde{A}_\mu = \sum_{\nu=0}^{3} (\lambda^{-1})_{\mu\nu}^T A_\nu \quad ,$$

then in normal vacuum[3] ($\eta = 1$)

$$\sum_{\mu\nu\alpha\beta} (-1)^{\delta_{\alpha 0} + \delta_{\beta 0}} \epsilon_{\alpha\beta\mu\nu} \tilde{\partial}_\alpha \tilde{A}_\beta d\tilde{q}_\mu \wedge d\tilde{q}_\nu = \sum_{\mu\nu\alpha\beta} (-1)^{\delta_{\alpha 0} + \delta_{\beta 0}} \epsilon_{\alpha\beta\mu\nu} \partial_\alpha A_\beta dq_\mu \wedge dq_\nu \quad ,$$

$$\tag{15}$$

and ω_f^* is Lorentz-invariant (Appendix 3).

From (9) and (14) one obtains

$$\det[\omega_f] = \frac{1}{\eta} \det[\omega_f^*] = (\mathbf{E} \cdot \mathbf{B})^2 \tag{16}$$

while (8), (13) yield

$$\omega_f^* \wedge \omega_f = (\eta \mathbf{E}^2 - \mathbf{B}^2) d\mathcal{V}^e \quad , \tag{17}$$

$d\mathcal{V}^e = dq_0 \wedge d\mathcal{V}$, $d\mathcal{V} = dq_1 \wedge dq_2 \wedge dq_3$.

The first 2-form $\omega_f = -d\theta_f$ is exact, so that $d\omega_f = 0$. This equality is equivalent to the first group of Maxwell equations [1]

$$\nabla \cdot \mathbf{B} = 0 \quad , \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad . \tag{18}$$

It is important to remark that although true magnetic charges have not been observed, low-lying excitations resembling free magnetic monopoles can

---

[3]A non-trivial, subluminal refractive index in vacuum, might be induced by quantum gravitational fluctuations [9].

arise as defects in spin ice [10, 11, 12, 13]. The equations of motion for a quasiparticle which carries both electric and magnetic charges are derived in Appendix 1. In the presence of a magnetic charge ω_f is only locally exact by the Poincaré lemma [14], and the Dirac's quantization condition can be retrieved as an integrality condition for $e\omega_f/hc$ with respect to any space-like, compact, oriented, 2-dimensional surface (Appendix 2).

If ρ denotes the free electric charge density, integrable over R^3, $\mathbf{j} = \rho\dot{\mathbf{q}}$, and

$$J = \rho d\mathcal{V} - \frac{1}{c}\mathbf{j} \cdot dq_0 \wedge d\mathbf{S} \tag{19}$$

is an invariant 3-form, then the second group of Maxwell equations can be written as[4]

$$d\omega_f^* = \mu_r J \quad . \tag{20}$$

Explicitly

$$d\omega_f^* = \eta\nabla \cdot \mathbf{E}d\mathcal{V} + (\eta\partial_0\mathbf{E} - \nabla \times \mathbf{B}) \cdot dq_0 \wedge d\mathbf{S} \quad , \tag{21}$$

such that (20) is equivalent to

$$\epsilon_r\nabla \cdot \mathbf{E} = \rho \quad , \quad \nabla \times \mathbf{B} = \frac{\mu_r}{c}(\mathbf{j} + \epsilon_r\frac{\partial\mathbf{E}}{\partial t}) \quad . \tag{22}$$

From (20) we also get $dJ = 0$, which provides the continuity equation

$$\partial_t\rho + \mathrm{div}\mathbf{j} = 0 \quad , \tag{23}$$

and $c\omega_f \wedge i_{\partial_0}d\omega_f^* = -\mu_r\mathbf{E} \cdot \mathbf{j}dq_0 \wedge d\mathcal{V}$, or explicitly

$$\eta\mathbf{E} \cdot \partial_0\mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{B} = -\frac{\mu_r}{c}\mathbf{E} \cdot \mathbf{j} \quad . \tag{24}$$

Replacing here $\mathbf{E} \cdot \nabla \times \mathbf{B} = -\mathrm{div}(\mathbf{E} \times \mathbf{B}) - \mathbf{B} \cdot \partial_0\mathbf{B}$ from (18), we obtain

$$\partial_t w + \mathrm{div}\mathbf{Y} + \mathbf{E} \cdot \mathbf{j} = 0 \quad , \tag{25}$$

where $w = (\epsilon_r\mathbf{E}^2 + \mathbf{B}^2/\mu_r)/2$ is the energy density of the field and $\mathbf{Y} = c\mathbf{E} \times \mathbf{B}/\mu_r$ the Poynting vector.

---

[4]A more general set of equations is provided by $cd\omega_f^* = \mu_r i_{X_{H^e}}J^e$, with $J^e = \rho^e d\mathcal{V}^e$ and $\rho^e(q^e, u)$ the extended electric charge density, integrable over R^4.

Worth noting is that when magnetic charges are included, the basic elements of the theory are the field 2-forms, rather than the local potentials $(\mathbf{A}, \mathsf{V})$, and the first group of Maxwell equations becomes $d\omega_f = -\mu_r J_m$, where J_m is the 3-form (19) for magnetic charges. Also, in vacuum, an integrality condition for ω_f^*/e, where e is a suitable constant, yields electric charge quantization (Appendix 2).

4 The characteristic vector field and photon dynamics

According to (16), in general the rank of the electromagnetic 2-forms ω_f, ω_f^* is not constant, and when $\mathbf{E} \cdot \mathbf{B} = 0$ both are degenerate. In the case $\mathbf{E} \cdot \mathbf{B} \neq 0$, ω_f is symplectic as it is closed by definition, while by (20), ω_f^* is closed only if $J = 0$.

Let us consider the common characteristic bundle P_f over space-time,

$$P_f = \{ V \in T\mathsf{R}^4 / i_V \omega_f = i_V \omega_f^* = 0 \} \quad . \tag{26}$$

Taking V of the form $V = V_0 \partial_0 + \mathbf{V} \cdot \nabla$, $i_V \omega_f = 0$ yields

$$\mathbf{E} \cdot \mathbf{V} = 0 \quad , \quad V_0 \mathbf{E} = -\mathbf{V} \times \mathbf{B} \quad , \tag{27}$$

while from $i_V \omega_f^* = 0$ one obtains

$$\mathbf{B} \cdot \mathbf{V} = 0 \quad , \quad V_0 \mathbf{B} = \eta \mathbf{V} \times \mathbf{E} \quad . \tag{28}$$

The equations (27), (28) hold also in inhomogeneous media, and have a solution $V \neq 0$ only if $\mathbf{E} \perp \mathbf{B}$ and $\mathbf{B}^2 = \eta \mathbf{E}^2 = \mu_r w$, when

$$\mathbf{V}^2 = \frac{1}{\eta} V_0^2 \quad , \quad \frac{\mathbf{V}}{V_0} = \frac{\mathbf{Y}}{cw} \quad . \tag{29}$$

Let us consider

$$\mathbf{E} = k_0 \mathbf{P}_\varphi \quad , \quad \mathbf{B} = \mathbf{k} \times \mathbf{P}_\varphi \quad , \tag{30}$$

where $\mathbf{P}_\varphi$ provides the polarization,

$$k_0 = -\partial_0 \varphi \quad , \quad \mathbf{k} = \nabla \varphi \quad , \tag{31}$$

and $\varphi(q_0, \mathbf{q})$ is the phase function. In this case

$$\omega_f = -\mathbf{P}_\varphi \cdot d\varphi \wedge d\mathbf{q} \quad , \tag{32}$$

such that $i_V \omega_f = 0$ if $i_V d\varphi = 0$, or

$$V_0 k_0 = \mathbf{V} \cdot \mathbf{k} \quad . \tag{33}$$

This equality, (29) and

$$\frac{\mathbf{k}}{k_0} = \eta \frac{\mathbf{V}}{V_0} \quad , \tag{34}$$

obtained from (28), yield

$$\mathbf{k}^2 = \eta k_0^2 \quad , \tag{35}$$

or, in terms of φ, the eikonal equation

$$(\nabla \varphi)^2 = \eta (\partial_0 \varphi)^2 \quad . \tag{36}$$

If $V \neq 0$ and $\mathbf{E}$, $\mathbf{B}$ are of the form (30), then in a non-dispersive, transparent medium ($\mathbf{j} = 0$), the coupled equations (25), (36) ensure an extremum for the action integral

$$\mathcal{A}[n, \varphi] = -\int d^4q \; n[\partial_t \varphi + \frac{c}{\sqrt{\eta}}|\nabla \varphi|] \tag{37}$$

with respect to the functional variations of the "photon density" $n(q_0, \mathbf{q})$ and $\varphi(q_0, \mathbf{q})$. Thus, $\delta_n \mathcal{A} = 0$ yields (36) in the form

$$\partial_t \varphi = -\frac{c}{\sqrt{\eta}}|\nabla \varphi| \quad , \tag{38}$$

while $\delta_\varphi \mathcal{A} = 0$ provides

$$\partial_t n + \mathrm{div}[n\frac{c}{\sqrt{\eta}} \frac{\nabla \varphi}{|\nabla \varphi|}] = 0 \quad . \tag{39}$$

In the stationary case $-\partial_t \varphi = ck_0 \equiv \omega$ is a constant, $|\nabla \varphi| = |\mathbf{k}| = \sqrt{\eta} k_0$, and (39) becomes

$$\partial_t n\omega + \mathrm{div}[n\frac{c^2}{\eta}\nabla \varphi] = 0 \quad . \tag{40}$$

8

By multiplication with $\hbar$, considered as a dimensional factor converting φ into the "mechanical" action $S = \hbar\varphi$, this equation becomes (25) with

$$w = n\hbar\omega \quad , \quad \mathbf{Y} = n\frac{c^2}{\eta}\hbar\mathbf{k} \quad , \tag{41}$$

up to additive constants. Worth noting, (38), (39) can also be expressed as an infinite-dimensional Hamiltonian system $i_{X_{\mathcal{H}}}\hat{\omega} = \mathsf{d}\mathcal{H}$, where [15]

$$\hat{\omega} = \int d\mathcal{V}\mathsf{d}n \wedge \mathsf{d}\varphi \quad , \quad \mathcal{H}[n,\varphi] = \int d\mathcal{V}\frac{c}{\sqrt{\eta}}n|\nabla\varphi| \tag{42}$$

and n, φ are the conjugate variables. Single photons in inhomogeneous media therefore appear as classical particles with the canonical coordinates $(\mathbf{q}, \mathbf{p} = \hbar\mathbf{k})$, and the Hamilton function $h_{sp}(\mathbf{q}, \mathbf{p}) = c|\mathbf{p}|/\sqrt{\eta(\mathbf{q})}$. The equations describing their motion along the light rays take in this case the form considered before in [16],

$$\dot{\mathbf{q}} = \nabla_{\mathbf{p}}h_{sp} = \frac{c}{\sqrt{\eta}}\frac{\mathbf{p}}{|\mathbf{p}|} \quad , \quad \dot{\mathbf{p}} = -\nabla_{\mathbf{q}}h_{sp} = h_{sp}\nabla\ln\sqrt{\eta} \quad . \tag{43}$$

The same equations can be obtained using (2), with an extended Hamiltonian $H_{sp}^e \equiv -c\sqrt{V_0^2 - \eta\mathbf{V}^2} = -c\sqrt{p_0^2 - \mathbf{p}^2/\eta}$ defined in terms of the scalar $V_0 k_0 - \mathbf{V}\cdot\mathbf{k}$, normalized by $V_0 = \hbar k_0 = -p_0$.

5   Summary and conclusions

The geometric description of the elctromagnetic field using 2-forms on the space-time manifold arises in relativistic particles dynamics, or charge quantization.

In this work have been considered two field 2-forms, in a relationship of duality. The first 2-form $\omega_f = -d\theta_f$ is exact in the absence of the magnetic monopoles, and in Section 2 it was used to describe the motion of an electric charge as a Hamiltonian flow on the extended phase-space. The dual form was defined in Section 3 in terms of the field components and the refractive index of the medium. It is shown that this form modifies the symplectic potential of the extended phase-space for a magnetic charge, providing the "electric Lorentz force" (Appendix 1). The exterior derivatives of the two

forms yield the two groups of Maxwell equations, while charge quantization can be introduced using specific integrality conditions (Appendix 2). In Section 4 the electromagnetic energy density and Poynting vector are related to the common characteristic vector (V) of the dual 2-forms. By the dependence on the refractive index, this vector and the "wave-vector" (k), derived from the phase function, resemble the energy-momentum 4-vectors of Abraham, respectively of Minkowski. It is shown that the coupled energy-density continuity equation and the eikonal equation can be described as a classical, infinite-dimensional Hamiltonian system, with the photon density and the phase function as conjugate variables. Single photons appear as classical particles having as Hamiltonian a function $h_{sp}(\mathbf{q}, \mathbf{p}) = \dot{\mathbf{q}} \cdot \mathbf{p}$, bilinear in velocity and momentum. Formally, one can also define an extended photon Hamiltonian, but further work is necessary to understand its significance, as in the limit of vanishing rest mass the universal time is not a suitable parameter.

6   Appendix 1: the equations of motion for electric/magnetic charges

The equations of motion in vacuum for a relativistic massive particle which carries beside the electric charge q_e, a magnetic charge q_m, can be obtained replacing ω_0^e in (2) by

$$\omega^e = \omega_0^e + \frac{q_e}{c}\omega_f + \frac{q_m}{c}\omega_f^* \quad . \tag{44}$$

In this case (12) are modified by the "electric Lorentz force" $-q_m\mathbf{v} \times \mathbf{E}/c$ and the magnetic field force $q_m\mathbf{B}$, such that one obtains

$$\dot{\mathbf{p}} = \frac{q_e}{c}\mathbf{v} \times \mathbf{B} - \frac{q_m}{c}\mathbf{v} \times \mathbf{E} + q_e\mathbf{E} + q_m\mathbf{B} \quad , \quad \dot{\mathcal{E}} = (q_e\mathbf{E} + q_m\mathbf{B}) \cdot \mathbf{v} \quad . \tag{45}$$

7   Appendix 2: the charge quantization

Let us consider the monopole vector field defined on $\mathsf{R}^3 - \{\mathbf{n}\mathsf{R}_-\}$,

$$\mathbf{G}_n(\mathbf{r}) = \frac{a}{r}\frac{\mathbf{n} \times \mathbf{e}_r}{1 + \mathbf{n} \cdot \mathbf{e}_r}$$

where a is a constant, $\mathbf{n}$ is a fixed unit vector, and

$$\mathbf{e}_r \equiv \frac{\mathbf{r}}{r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \ ,$$

with r, θ, φ the usual spherical coordinates of the position vector $\mathbf{r}$ in R^3.

Because $\nabla \times \mathbf{G_n} = a\mathbf{e}_r/r^2$ independently of $\mathbf{n}$, a suitable set of local 1-forms $\alpha_\mathbf{n} = \mathbf{G_n} \cdot d\mathbf{r}$ defines a symplectic form $\tilde{\omega}$ on the unit sphere S^2,

$$\tilde{\omega}|_{U_\mathbf{n}} \equiv d\alpha_\mathbf{n} = (\nabla \times \mathbf{G_n}) \cdot d\mathbf{S} = a\sin\theta d\theta \wedge d\varphi \ ,$$

where the open set $U_\mathbf{n} = \mathsf{S}^2 - \{P_{-\mathbf{n}}\}$ is the domain of $\alpha_\mathbf{n}$, obtained by removing from S^2 the "pole" $P_{-\mathbf{n}}$ located at $\mathbf{r} = -\mathbf{n}$. Thus, if $\mathbf{n}$, $\mathbf{n}'$ are two distinct unit vectors, $\tilde{\omega} = d\alpha_\mathbf{n} = d\alpha_{\mathbf{n}'}$ on $U_\mathbf{n} \cap U_{\mathbf{n}'}$, and the 1-form $\alpha_\mathbf{n} - \alpha_{\mathbf{n}'} \equiv d\Phi_{\mathbf{nn}'}$ is exact. For instance, if $\mathbf{n} = \mathbf{k}$ and $\mathbf{n}' = -\mathbf{k}$, with $\mathbf{k} \equiv (0,0,1)$, along the Z-axis, then

$$\alpha_\mathbf{k} = a(1 - \cos\theta)d\varphi \ , \quad \alpha_{-\mathbf{k}} = a(-1 - \cos\theta)d\varphi \tag{46}$$

and

$$\alpha_\mathbf{k} - \alpha_{-\mathbf{k}} = 2ad\varphi \ , \tag{47}$$

while by taking $\mathbf{n}' = \mathbf{i} \equiv (1,0,0)$, along the X-axis,

$$\Phi_{\mathbf{ki}} = a[\varphi + \arctan(\sin\varphi\tan\theta) + \arctan(\cot\varphi\cos\theta)] \ . \tag{48}$$

In general, if α_i, α_j, defined on the open subsets $U_i, U_j \subset M$, are local 1-forms associated to a connection in a complex line bundle L over the manifold M, with curvature $\tilde{\omega}$, then on $U_i \cap U_j$

$$\alpha_i - \alpha_j = \frac{1}{2\pi i}\frac{dc_{ij}}{c_{ij}} \ ,$$

where $c_{ij} : U_i \cap U_j \mapsto \mathsf{C}$ are the transition functions [17]. For the unit circle subbundle of L, c_{ij} become $c_{ij}^1 : U_i \cap U_j \mapsto \mathsf{U}(1)$ such that $c_{\mathbf{k}-\mathbf{k}}^1 = e^{4\pi i a\varphi}$ provided by (47) is well-defined only if $4\pi a = n$, $n \in \mathsf{Z}$. In this case, if $\gamma \subset U_\mathbf{k} \cap U_{-\mathbf{k}}$ is any curve on S^2, closed around the Z-axis, then

$$\int_{\mathsf{S}^2} \tilde{\omega} = \oint_\gamma (\alpha_\mathbf{k} - \alpha_{-\mathbf{k}}) = 4\pi a = n \ . \tag{49}$$

If the magnetic field in (44) is $\mathbf{B} = \mu_r q'_m \mathbf{e}_r / 4\pi r^2$, as expected for a point-like magnetic charge q'_m, and $\tilde{\omega}$ is identified with the space-like term $q_e \mathbf{B} \cdot d\mathbf{S} / hc$ of ω^e / h, then in vacuum $a = q_e q'_m / 4\pi hc$ and (49) yields the Dirac's quantization condition

$$q_e q'_m = nhc \quad , \quad n \in \mathsf{Z} \quad . \tag{50}$$

One should note though that formally, the electric field $\mathbf{E} = q'_e \mathbf{e}_r / 4\pi \epsilon_r r^2$ produced by a point-like electric charge q'_e can also be expressed locally as $\mathbf{E} = q'_e \nabla \times \mathbf{G_n} / 4\pi \epsilon_r a$. In such a case, in vacuum, an integrality condition for the term $q_m \mathbf{E} \cdot d\mathbf{S} / hc$ of ω^e / h reproduces (50) in the form $q'_e q_m = nhc$, but if $\tilde{\omega}$ is identified with the space-like term in ω^*_f / e then $a = q'_e / 4\pi e$ and (49) provides electric charge quantization, $q'_e = ne$, $n \in \mathsf{Z}$.

8   Appendix 3: the Lorentz-invariance of ω^*_f

To prove (15), let $\lambda \in SO(1,3)^*$ be a Lorentz transformation,

$$\lambda^T \hat{g} \lambda = \hat{g} \quad , \quad \hat{g} = diag[-1, 1, 1, 1] \quad , \tag{51}$$

such that

$$\tilde{q}_\mu = \sum_{\nu=0}^{3} \lambda_{\mu\nu} q_\nu \quad , \quad \tilde{A}_\mu = \sum_{\nu=0}^{3} (\lambda^{-1})^T_{\mu\nu} A_\nu \quad .$$

If we denote $\tilde{\partial}_\mu \equiv \partial / \partial \tilde{q}_\mu$, then

$$\tilde{\partial}_\mu = \sum_{\nu=0}^{3} (\lambda^{-1})^T_{\mu\nu} \partial_\nu \quad ,$$

and the left-hand side of (15) takes the form

$$\sum_{\alpha\beta\mu\nu} (-1)^{\delta_{\alpha 0} + \delta_{\beta 0}} \epsilon_{\alpha\beta\mu\nu} \tilde{\partial}_\alpha \tilde{A}_\beta d\tilde{q}_\mu \wedge d\tilde{q}_\nu$$

$$= \sum_{\alpha\beta\mu\nu} \sum_{ijkl} (-1)^{\delta_{\alpha 0} + \delta_{\beta 0}} \epsilon_{\alpha\beta\mu\nu} (\lambda^{-1})^T_{\alpha i} (\lambda^{-1})^T_{\beta j} \lambda_{\mu k} \lambda_{\nu l} \partial_i A_j dq_k \wedge dq_l \quad .$$

As $\hat{g}^2 = I$, from (51) we also get $\hat{g}(\lambda^{-1})^T = \lambda \hat{g}$, or in terms of the matrix elements

$$(-1)^{\delta_{\alpha 0}} (\lambda^{-1})^T_{\alpha i} = \lambda_{\alpha i} (-1)^{\delta_{i0}} \quad ,$$

and the previous expression becomes

$$\sum_{\alpha\beta\mu\nu} \sum_{ijkl} \epsilon_{\alpha\beta\mu\nu} \lambda_{\alpha i} \lambda_{\beta j} \lambda_{\mu k} \lambda_{\nu l} (-1)^{\delta_{i0}+\delta_{j0}} \partial_i A_j dq_k \wedge dq_l \quad . \qquad (52)$$

However,

$$\sum_{\alpha\beta\mu\nu} \epsilon_{\alpha\beta\mu\nu} \lambda_{\alpha i} \lambda_{\beta j} \lambda_{\mu k} \lambda_{\nu l} = \epsilon_{ijkl} \det(\lambda) \quad ,$$

whyle $\det(\lambda) = 1$ because $\lambda \in SO(1,3)^*$, such that (52) becomes the right-hand side of (15).

References

[1] H. Weyl, *Gravitation und Elektrizität*, Sitz. Kön. Preuss. Akad. Wiss. Berlin (1918), p. 465

[2] P. A. M. Dirac, *Quantised singularities in the electromagnetic field*, Proc. R. Soc. A **133** 60 (1931)

[3] J. Śniatycki, *Geometric Quantization and Quantum Mechanics*, Springer, New York (1980)

[4] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press (1984), p. 140

[5] W. E. Baylis, *Electrodynamics, A Modern Geometric Approach*, Birkhäuser, Boston (1999), p. 106

[6] M. Grigorescu, *Energy and time as conjugate dynamical variables* , Can. J. Phys. **78** 959 (2000)

[7] M. Grigorescu, *Relativistic probability waves*, Rev. Roumaine Math. Pures Appl. **55** 131 (2010)

[8] W. J. Padilla, D. N. Basov, D. R. Smith, *Negative refractive index in metamaterials*, Materialstoday, **9** 28 (2006)

[9] J. Ellis, N. E. Mavromatos and D. V. Nanopoulos, *Derivation of a vacuum refractive index in a stringy space-time foam model*, Phys. Lett. B **665** 412 (2008)

[10] C. Castelnovo, R. Moessner and S. L. Sondhi, *Magnetic monopoles in spin ice*, Nature **451** 42 (2008)

[11] S. T. Bramwell, S. R. Giblin, C. Calder, R. Aldus, D. Prabhakaran and T. Fennell, *Magnetic charge transport*, Nature **461** 956 (2009)

[12] L. D. C. Jaubert and P. C. W. Holdsworth, *Signature of magnetic monopole and Dirac string dynamics in spin ice*, Nature Physics **5** 258 (2009)

[13] H. Kadowaki, N. Doi, Y. Aoki, Y. Tabata, T. J. Sato, J. W. Lynn, K. Matsuhira and Z. Hiroi, *Observation of magnetic monopoles in spin ice*, J. Phys. Soc. Jpn. **78** 103706 (2009)

[14] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, Benjamin, New York (1978), p. 118

[15] M. Grigorescu, *Classical probability waves*, Physica A **387** 6497 (2008)

[16] M. Marklund, *Radiation transport in diffractive media*, J. Phys. A **38** 4265 (2005)

[17] B. Kostant, *Quantization and Unitary Representations*, Lecture Notes in Mathematics, vol. 170, Springer, New York (1970)